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SEMIRINGS AND T_1 COMPACTIFICATIONS. I

BY

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ABSTRACT. With each infinite cardinal ω_μ is associated a topological semiring $\mathbf{F}_\mu$, whose underlying space is finite complement topology on the set of all ordinals less than ω_μ, and whose operations are the natural sum and natural product defined by Hessenberg. The theory of the semirings $C_\mu(X)$ of maps from a space X into $\mathbf{F}_\mu$ is developed in close analogy with the theory of the ring $C(X)$ of continuous real-valued functions; the analogy is not on the surface alone, but may be pursued in great detail. With each semiring a structure space is associated; the structure space of $C_\mu(X)$ for sufficiently large ω_μ will be the Wallman compactification of X. The classes of ω_μ-entire and ω_μ-total spaces, which are respectively analogues of realcompact and pseudocompact spaces, are examined, and an ω_μ-entire extension analogous to the Hewitt realcompactification is constructed with the property (not possessed by the Wallman compactification) that every map between spaces has a unique extension to their ω_μ-entire extensions. The semiring of functions of compact-small support is considered, and shown to be related to the locally compact-small spaces in the same way that the ring of functions of compact support is related to locally compact spaces.

1. Introduction. The study of the T_1 compactifications of a given space is an uncharted wilderness in which a few paths have been cleared. There have been no general techniques comparable in power and in scope to those techniques that have been developed for Hausdorff compactifications. The absence of such general methods is undoubtedly due to the extreme variety of T_1 compactifications that can exist. In contrast to the Hausdorff case there is no upper limit on their cardinality, and the topological properties that they may possess are practically unlimited.

In the theory of Hausdorff compactifications there are two spaces that play a fundamental role; the unit interval $\mathbf{I}$ and the real line $\mathbf{R}$. The role of $\mathbf{I}$ is as a universal embedding space; the compact Hausdorff spaces are the closed subspaces of products of $\mathbf{I}$. It is known that no space can serve this role for all compact T_1 spaces. The problem is appropriately reformulated in [HS1] so that a solution does exist; the resulting theory of universal spaces is best possible in an appropriate sense. The role of $\mathbf{R}$ is most clear in the theory of

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rings of continuous functions, in which its algebraic structure plays a primary role, and compactifications are studied as maximal ideal spaces of rings of continuous functions. It is this method of examining Hausdorff compactifications that is the model for the present method.

With each infinite cardinal (= initial ordinal) ω_μ is associated a topological semiring $\mathbf{F}_\mu$, whose underlying space is a finite complement space on the set of ordinals less than ω_μ. The theory of the semirings $C_\mu(X)$ of continuous functions from a space X into $\mathbf{F}_\mu$ bears a close analogy to the theory of the ring $C(X)$ of continuous real-valued functions. The analogy is not on the surface alone, but may be pursued in great detail.

With each semiring is associated a structure space (of m-ideals, which are maximal with respect to a natural property). For sufficiently large ω_μ the structure space of $C_\mu(X)$ will be the Wallman compactification of X. The relationship of the theory of maximal filters consisting of the zero sets of a semiring $C_\mu(X)$ to that of maximal m-ideals parallels in many ways the relationship of the theory of z-ultrafilters to that of maximal ideals in $C(X)$.

The most surprising results appear in connection with ω_μ-entire spaces, which are analogues of realcompact spaces, defined in terms of maximal m-ideals in $C_\mu(X)$ just as realcompact spaces are defined in terms of maximal ideals in $C(X)$. The property is closed hereditary, productive, and intersective. Much more significantly, it can be shown that the analogue of the process of Hewitt realcompactification leads to a minimal ω_μ-entire subspace $v_\mu X$ of the Wallman compactification; the significance is that, although Wallman compactification is functorial only for restrictive classes of maps, and epireflective only under still further restriction, the assignment $X \to v_\mu X$ is an epireflection from the category of spaces and maps to the category of ω_μ-entire spaces and maps.

The final section examines the T_1 analogue of locally compact spaces, the locally compact-small spaces. The ring of functions of compact-small support plays here a role analogous to the role of functions with compact support in $C(X)$; more specifically, the analogy is with the functions of small support in $C(X)$.

In a later paper a complete theory in terms of which *any* T_1 compactification of a space may be obtained algebraically from an appropriate $C_\mu(X)$ will be exhibited. The method is based upon an extension of the notion of a homomorphism.

There are two fundamental reasons for studying T_1 spaces through semirings. The first is that such study leads to the examination and study of interesting classes of spaces whose specific properties have not previously been noted; this is the case for the ω_μ-entire spaces and the locally compact-small spaces, for example. This approach sheds light upon the map extension problem, which is very difficult in the absence of regularity and in the absence of the Hausdorff condition. Another justification for the study is that it may also shed light upon the theory of rings of continuous functions by reverse analogy. The more set-theoretical arguments used in the study of $C_\mu(X)$ help us to understand which arguments about rings $C(X)$ involve in an essential manner the properties of the real line, and which are more purely a result only of the abstract setting of the results.

Throughout the remainder of this paper the term *space* shall mean a T_1 topological space and the term *map* shall mean a continuous function between spaces. The symbol crdn X represents the cardinal number of the set X.

Since there will be no occasion to do so in the body of the paper, it is appropriate to refer now to Hewitt's epoch-making paper [HW]. It has been the primary influence on the present work, which may best be conceived as an attempt to begin to realize for T_1 spaces the ideal set forth by Hewitt "that by utilizing appropriate algebraic techniques ... some progress may be made toward establishing general methods for the solution of topological problems." The theory of $C_\mu(X)$ as herein developed is only a pale shadowing of the theory of $C(X)$ as developed in [HW] and [GJ]; perhaps the simple structure of $\mathbf{F}_\mu$ in comparison with the rich structure of $\mathbf{R}$ makes only a pale shadowing possible. Certainly a vigorous examination of the properties of the ordered topological semirings $\mathbf{F}_\mu$ and $C_\mu(X)$ will strengthen this shadowing and should be made. None is attempted here.

2. The topological semirings $\mathbf{F}_\mu$. A *semiring* is a set S with two binary associative commutative operations, addition and multiplication, with identities 0 and 1 respectively, such that multiplication distributes over addition. A *topological semiring* is a semiring S with a topology under which addition and multiplication are continuous as maps from $S \times S$ into S.

Let ω_μ be an infinite cardinal, and let $\mathbf{F}_\mu$ be the space of all ordinals less than ω_μ taken with finite complement topology. There are very natural operations on $\mathbf{F}_\mu$ under which it becomes a topological semiring. Recall ([SP, p. 323] or [KE, p. 109]) that every ordinal $\neq 0$ has a unique *natural representation* in the form $\omega_0^{\xi_1} m_1 + \cdots + \omega_0^{\xi_r} m_r$, where $\xi_1, \cdots, \xi_r$ are strictly decreasing ordinals and $m_1, \cdots, m_r$ are positive integers. The *natural sum* and the *natural product* are defined just as for polynomials ([SP, p. 366] or [KE p. 109]), with the stipulation that addition of exponents during multiplication is natural addition. These are commutative associative operations, multiplication distributes over addition, and there are identities 0 and 1. Also the degree of a nonzero ordinal is defined as the highest exponent that appears in its representation.

An initial ordinal ω_μ for $\mu > 0$ is an ϵ-number [SP, p. 397], which means that $\omega_0^{\omega_\mu} = \omega_\mu$. It follows that $\mathbf{F}_\mu$ can be thought of as the collection of ordinals of degree less than ω_μ (with 0 added). It is now readily shown that $\mathbf{F}_\mu$ is a semiring under these operations. It is worth noting that $\mathbf{F}_\mu$ may be embedded into an integral domain, by extending the operations to allow arbitrary integer coefficients. The field of quotients of this integral domain has been examined by Sikorski [SI1],[SI2]; see [HS2] for other references and applications.

To see that $\mathbf{F}_\mu$ is a topological semiring observe that the equations $a + b = c$ and $ab = c$ have only finitely many solutions [KE, p. 110]; thus the inverse image of every finite set is finite, under addition or multiplication.

It should be noted that the usual ordering of $\mathbf{F}_\mu$ is compatible with the algebraic operations, so that $\mathbf{F}_\mu$ is an *ordered topological* semiring. However, the order structure seems to play a lesser role in the theory herein developed, perhaps due to the nonarchimedean character of $\mathbf{F}_\mu$ when $\mu \neq 0$.

3. **The function semirings $C_\mu(X)$.** Given any space X the collection $C_\mu(X)$ of maps from X into $\mathbf{F}_\mu$ forms a semiring. Such maps must be examined in connection with the *associated partition* $\{f^\leftarrow(f(x)): x \in X\}$.

3.1. A function $f: X \to \mathbf{F}_\mu$ is a map if and only if the associated partition consists of closed sets.

A partition of the space X into no more than ω_μ closed subsets is called an *ω_μ-partition* of X.

3.2. Every ω_μ-partition of a space is the associated partition of some $f \in C_\mu(X)$. The associated partitions of f and g are equal if and only if there is a permutation h of $\mathbf{F}_\mu$ such that $hf = g$.

The concept of fréchet character introduced in [HS1] is an invariant similar to cardinality in that its particular value is not usually of primary interest, but its consideration is essential in many contexts.

The *partition character* prtn B of a closed subset B of X is the least cardinal $\mathfrak{m}$ for which B belongs to a partition of X into no more than $\mathfrak{m}$ closed sets; the *partition character* prtn $\mathfrak{B}$ of a collection $\mathfrak{B}$ of closed sets is the supremum of prtn B over all $B \in \mathfrak{B}$. The *fréchet character* fchr X of the space X is the least partition character of a closed base for X. The empty space is assigned fréchet character 0.

3.3. A space has fréchet character 1 if and only if it is a one-point space.

3.4. If crdn $X > 1$ and fchr X is finite then fchr $X = 2$; equivalently the space has a base of open-and-closed sets.

3.5. If $\mathfrak{B}$ is any collection of closed subsets of X then prtn $\mathfrak{B} \leq$ crdn X. In particular, fchr $X \leq$ crdn X.

3.6. For each ω_μ, fchr $\mathbf{F}_\mu = \omega_\mu$.

The connection between fréchet character and $C_\mu(X)$ is given in the following results.

3.7. If $B \subset X$ is closed then prtn $B \leq \omega_\mu$ if and only if there is a map χ_B: $X \to \mathbf{F}_\mu$ such that $B = \chi_B^{\leftarrow}(0)$.

3.8. A space has fréchet character $\leq \omega_\mu$ if and only if it has the weak topology of $C_\mu(X)$; equivalently, it is homeomorphic to some subspace of a product of copies of $\mathbf{F}_\mu$.

In the computation of fréchet character the following results are essential.

3.9. For each space X, fchr $X = \sup \{\text{fchr } A: A \subset X\}$.

3.10. For each family $\{X_\alpha\}$ of spaces, fchr $\Pi X_\alpha = \sup \{\text{fchr } X_\alpha: X_\alpha \in \{X_\alpha\}\}$.

For a completely regular space the zero set of a function in $C(X)$ can be considered as the zero set of a function in $C_\mu(X)$, where $\omega_\mu = \mathfrak{c}$ is the cardinality of $\mathbf{R}$; thus the following result.

3.11. The fréchet character of a completely regular space is not greater than $\mathfrak{c}$.

3.12. **Remark.** It is not difficult to see that the fréchet character of a completely regular space is not greater than the fréchet character of $\mathbf{R}$. The compact connected space $\mathbf{R}$ has no ω_0-partitions [BU, p. 213], thus fchr $\mathbf{R} \geq \omega_1$. Whether equality holds is unkown to the author; it seems likely that fchr $\mathbf{R}$ = crdn $\mathbf{R}$.

4. **Ideals and homomorphisms.** Some properties of ideals and homomorphisms in semirings that are frequently useful are collected here. The present paper makes essential use only of m-ideals, which are discussed in §5.

An *ideal* in a semiring S is a subset I such that $S(I + I) \subset I$. A *congruence* on S is an equivalence relation $\equiv$, such that if $a \equiv b$ and $c \equiv d$ then $a + c \equiv b + d$ and $ac = bd$. A *homomorphism* from the semiring S to the semiring T is a function $\Psi: S \to T$ such that $\Psi(a + b) = \Psi(a) + \Psi(b)$, $\Psi(ab) = \Psi(a)\Psi(b)$, $\Psi(0) = 0$, and $\Psi(1) = 1$. An *isomorphism* is a bijective homomorphism.

4.1. If I is an ideal in S then the relation $\equiv_I$, given by $a \equiv_I b$ if there are $c, d \in I$ with $a + c = b + d$, is a congruence.

4.2. If $\equiv$ is a congruence on S and $S/\equiv$ is the set of equivalence classes, then $S/\equiv$ is a semiring under $a/\equiv + b/\equiv = (a+b)/\equiv$, $(a/\equiv)(b/\equiv) = (ab)/\equiv$, with additive and multiplicative identities $0/\equiv$ and $1/\equiv$, and the projection $a \to a/\equiv$ is a semiring homomorphism. Also $0/\equiv$ is an ideal in S.

4.3. If $\Psi: S \to T$ is a homomorphism, then the relation $\equiv_\Psi$, given by $a \equiv_\Psi b$ if $\Psi(a) = \Psi(b)$, is a congruence. The correspondence $a/\equiv \to \Psi(a)$ is an isomorphism between $S/\equiv$ and the subsemiring $\Psi[S]$ of T.

The bijective correspondence that obtains in rings between ideals and congruences need not hold in semirings, and thus the correspondence between ideals and surjective homomorphisms does not hold either. For each congruence there is the *associated ideal* $0/\equiv$. Now order congruences by inclusion as subsets of $S \times S$, and define a *minicongruence* to be a congruence $\equiv$ that is minimum among congruences having $0/\equiv$ as associated ideal.

4.4. The correspondence $I \to \equiv_I$ is a bijection between ideals and mini-congruences.

Proof. The only portion that requires proof is to show that $\equiv_I$ is a minicongruence. Suppose to this end that $0/\equiv\; = I$, and $a \equiv_I b$. Then there are $c, d \in I$ such that $a + c = b + d$. Now $c \equiv 0$ and $d \equiv 0$, thus $a \equiv a + c = b + d \equiv b$.

In terms of homomorphisms the analogues of minicongruences are minihomomorphisms. The homomorphism $\phi: S \to T$ is a *minihomomorphism* if whenever $\Psi: S \to R$ and $\Psi^{\leftarrow}(0) \supset \phi^{\leftarrow}(0)$ then there is $\theta: T \to R$ with $\theta\phi = \Psi$. The following result is easily shown.

4.5. The surjective homomorphism Ψ is a minihomomorphism if and only if $\equiv_\Psi$ is a minicongruence.

5. **Maximal m-ideals.** The special type of ideal now to be introduced finds essential application in the representation theorem for compact spaces.

An *m-unit* in the semiring S is an element a that is multiplicatively cancellable; that is, if $ab = ac$ then $b = c$, for all $b, c \in S$. If S is a ring the m-units are the non-zero-divisors (the *regular* elements of [ZS]). An *m-ideal* is an ideal which does not contain an m-unit.

5.1. For each $a, b \in S$ the product ab is an m-unit if and only if both a and b are m-units.

5.2. The ideal Sa generated by $a \in S$ is an m-ideal if and only if a is not an m-unit.

5.3. **Remark.** The ideal generated by two elements that are not m-units need not be an m-ideal; for example the sum of two zero-divisors in a ring may be equal to 1.

The following consequence of Zorn's lemma will be needed below.

5.4. Every m-ideal is contained in a maximal m-ideal.

5.5. If I is a maximal m-ideal and $a, b \in S$ then $ab \in I$ if and only if $a \in I$ or $b \in I$.

6. **The structure space.** Every semiring has a space of maximal m-ideals, in close analogy with the Stone structure space of a ring.

Let $\mathfrak{M}$ be the collection of maximal m-ideals in S and for each $a \in S$ define $\mathfrak{M}_a = \{M \in \mathfrak{M}: a \in M\}$. It follows from 5.5 that $\mathfrak{M}_a \cup \mathfrak{M}_b = \mathfrak{M}_{ab}$ for each $a, b \in S$, and thus $\{\mathfrak{M}_a: a \in S\}$ may be taken as closed base for a topology on $\mathfrak{M}$. The usual methods of proof [GJ, 7M] may be applied to demonstrate the following:

6.1. The set $\mathfrak{M}$ with $\{\mathfrak{M}_a : a \in S\}$ as closed base is a compact space.

6.2. **Remark.** It is of interest to construct the structure space of maximal m-ideals in the case when S is a ring. Since the collection of non-zero-divisors need not form a ring, the space $\mathfrak{M}$ need not be the (ordinary) maximal ideal space of any subring of S. However, it can be shown without much difficulty (using [ZS, §10, pp. 223–224]) that the space $\mathfrak{M}$ is homeomorphic to the maximal ideal space of the total quotient ring S [ZS, p. 44].

6.3. **Remark.** In the special case when S is a ring $C(X)$ of continuous real-valued functions, the total quotient ring is considered in [FGL]; the m-units in $C(X)$ are precisely those functions $f \in C(X)$ for which $\{x: f(x) \neq 0\}$ is dense in X. Equivalently, the zero set of f has empty interior.

7. **The function semirings** $C_\mu(X)$. The specific nature of the semirings $C_\mu(X)$ will now be taken into consideration. The algebraic properties of the elements of such semirings are in intimate relationship with their zero sets, just as in the rings $C(X)$.

If $f: X \to F_\mu$ define $Zf = \{x: f(x) = 0\}$; set $Z_\mu(x) = \{Zf: f \in C_\mu(x)\}$. The set-theoretic operations involving zero sets of functions in $C_\mu(X)$ are readily described algebraically in similar but simpler terms than for a ring $C(X)$.

7.1. If $f, g \in C_\mu(X)$ then $Z(f+g) = Zf \cap Zg$ and $Z(fg) = Zf \cup Zg$.

Before describing m-units in $C_\mu(X)$ algebraically a preliminary definition must be made. Let $\phi_1: \mathbf{F}_\mu \to \mathbf{F}_\mu$ be defined by $\phi_1(0) = 1$, $\phi_1(\delta) = \delta$ for $\delta \neq 0$. Then ϕ_1 is continuous and thus for each $f \in C_\mu(X)$ the function $f_1 = \phi_1 f \in C_\mu(X)$.

7.2. For each $f \in C_\mu(X)$ $ff_1 = f^2$.

7.3. The map $f \in C_\mu(X)$ is an m-unit if and only if $Zf = \varnothing$.

Proof. Suppose $Zf = \varnothing$. If $gf = hf$ for $g, h \in C_\mu(X)$ then for each $x \in X$, $g(x)f(x) = h(x)f(x)$; since $f(x) \neq 0$ and $\mathbf{F}_\mu$ is embeddable in an integral domain then $g(x) = h(x)$. Thus $g = h$, so f is an m-unit.

Conversely, if $f \in C_\mu(X)$ is an m-unit then since $ff_1 = f^2$ it follows that $f = f_1$ and thus $Zf = \varnothing$.

7.4. **Remark.** In $C(X)$ the functions with empty zero sets are the units; as already mentioned in 6.3 the m-units are the functions with nowhere dense zero sets.

For each $x \in X$ define $M_\mu^x = \{f \in C_\mu(X): f(x) = 0\}$.

7.5. The collection M_μ^x is a maximal m-ideal in $C_\mu(X)$.

Proof. Clearly M_μ^x is an m-ideal. Now if $g \in C_\mu(X)$ and $g(x) \neq 0$ then $g(x) = \delta$ for some $\delta \in \mathbf{F}_\mu - \{0\}$. If $\phi: \mathbf{F}_\mu \to \mathbf{F}_\mu$ interchanges δ with 0 then $f = \phi g \in M_\mu^x$ and since $Zf \cap Zg = \varnothing$ then, by 7.1 and 7.3, $f + g$ is an m-unit. It now follows that M_μ^x is a maximal m-ideal.

An algebraic description of the inclusion relation for zero sets will now be given.

7.6. The following are equivalent for $f, g \in C_\mu(X)$.

(a) $Zf \subset Zg$.

(b) Every maximal m-ideal that contains f also contains g.

(c) There is an m-unit j such that $jg = fg$.

(d) $f_1 g = fg$.

(e) For each $h, k \in C_\mu(X)$ if $fh = fk$ then $gh = gk$.

Proof. It follows as in the first part of the proof of 7.3 that (a) implies (e). From 7.2 it follows that (e) implies (d) and since f_1 is an m-unit then (d) implies (c). Now if (c) holds and M is a maximal m-ideal with $f \in M$ then $fg = jg \in M$, and from 5.5 and the fact that j is an m-unit it follows that $g \in M$. That (b) implies (a) is immediate from 7.5.

7.7. **Remark.** The inclusion relationship of zero sets has an algebraic description in $C(X)$: $Zf \subset Zg$ if and only if every maximal ideal that contains f also contains g [GJ, 4A].

The following simply established result is frequently useful.

7.8. A closed subset B of X belongs to the associated partition of some $f \in C_\mu(X)$ if and only if it belongs to $Z_\mu(X)$.

The sets A and B are *separated* by $f \in C_\mu(X)$ if $A \subset f^{\leftarrow}(0)$ and $B \subset f^{\leftarrow}(1)$. The following result parallels [GJ, 1.15] and is used for the same purposes. The present result is much more purely set-theoretic in nature, however.

7.9. Two sets are separated by some $f \in C_\mu(X)$ if and only if they are contained in disjoint zero sets of $C_\mu(X)$.

Proof. By 7.8 two separated sets are contained in disjoint zero sets. Conversely, if $A \subset Zg$ and $B \subset Zh$ with $Zg \cap Zh = \emptyset$, form the ω_μ-partition $\{Zg, Zh\} \cup \{g^{\leftarrow}(\delta) \cap h^{\leftarrow}(\epsilon): \delta \neq 0 \neq \epsilon,\ \delta, \epsilon \in \mathbf{F}_\mu\}$, and define $f: X \to \mathbf{F}_\mu$ by $f(x) = 0$ for $x \in Zg$, $f(x) = 1$ for $x \in Zh$, and by making arbitrary distinct choices of elements of $\mathbf{F}_\mu \neq 0, 1$ for the other members of the partition.

The following result shows that the base $\{Mf: f \in C_\mu(X)\}$ of closed sets in the maximal m-ideal space of $C_\mu(X)$ is closed under finite intersection as well as finite union.

7.10. If $f, g \in C_\mu(X)$ then $\mathfrak{M}_f \cap \mathfrak{M}_g = \mathfrak{M}_{f+g}$.

Proof. If M is a maximal m-ideal in $C_\mu(X)$ and $M \in \mathfrak{M}_f \cap \mathfrak{M}_g$ then $f, g \in M$, thus $f + g \in M$ and so $M \in \mathfrak{M}_{f+g}$. If $M \notin \mathfrak{M}_f \cap \mathfrak{M}_g$ then say $M \notin \mathfrak{M}_f$, so $f \notin M$, and thus there are $k \in M$ and $h \in C_\mu(X)$ such that $k + hf$ is an m-unit. By 7.3 it follows that $Z(k + hf) = \emptyset$. Now from 7.1 it follows that $Zk \cap (Zh \cup Zf) = \emptyset$, and thus $Zk \cap Zf = \emptyset$, so $Zk \cap Zf \cap Zg = Zk \cap Z_{f+g} = \emptyset$. Thus by 7.3 $f + g + k$ is an m-unit and therefore $f + g + k \notin M$, so $f + g \notin M$, that is $M \notin \mathfrak{M}_{f+g}$.

8. m-**ideals and** Z_μ-**filters.** There is a duality between a certain class of m-ideals and a certain class of filters that follows closely the duality between z-ideals and z-filters in a ring $C(X)$.

If I is any subset of $C_\mu(X)$ one can form the collection $Z[I] = \{Zf: f \in I\}$ of zero sets of functions in I. If Z is any collection of zero sets of functions in $C_\mu(X)$ one can form the collection $I[Z] = \{f: Zf \in Z\}$ of functions whose zero set belongs to Z. The collection $Z_\mu(X) = Z[C_\mu(X)]$ is a lattice under the set-theoretic operations; a Z_μ-*filter* is a filter on this lattice. The proofs of the following results follow those of the corresponding results in [GJ, Chapter 2].

8.1. The ideal I is an m-ideal in $C_\mu(X)$ if and only if $Z[I]$ is a Z_μ-filter.

8.2. If F is a Z_μ-filter on X then $I[F]$ is an m-ideal in $C_\mu(X)$. If $F = Z[I]$ for any ideal I then $I \subset I[F]$.

8.3. The correspondence $F \to I[F]$ is a bijection between maximal Z_μ-filters and maximal m-ideals.

8.4. **Remark.** The key to the above results is the set-theoretic description in 7.3 of the m-units as the functions with empty zero set. Thes 8.1, 8.2 and 8.3 will hold in any subsemiring of $C_\mu(X)$ in which this description of m-units holds.

The most important class of ideals in $C_\mu(X)$ is the Z_μ-*ideals*, which consists of ideals I for which $I = I[Z[I]]$. Clearly, by 7.6 such ideals have an algebraic description.

8.5. The ideal I is an m-ideal if and only if $I[Z[I]]$ is an m-ideal.

8.6. Every maximal m-ideal is a Z_μ-ideal.

The most important property of Z_μ-ideals is the following set-theoretic description of the associated congruence.

8.7. Let I be a Z_μ-ideal. Then $f \equiv_I g$ if and only if there is $h \in I$ such that $f(x) = g(x)$ whenever $h(x) = 0$.

Proof. Suppose I is any ideal and $f \equiv_I g$. Then there are $m, n \in I$ such that $f + m = g + n$. Now $h = m + n \in I$ and $f(x) = g(x)$ whenever $h(x) = 0$.

Conversely suppose I is a Z_μ-ideal and there is $h \in I$ such that $f(x) = g(x)$ whenever $h(x) = 0$. Let $m(x) = g(x)$ if $h(x) \neq 0$ and $=0$ otherwise; let $n(x) = f(x)$ if $h(x) \neq 0$ and $=0$ otherwise. Now $m^{\leftarrow}(0) = h^{\leftarrow}(0) \cup g^{\leftarrow}(0)$ and if $a \neq 0$, $m^{\leftarrow}(a) = g^{\leftarrow}(a)$. Thus m is continuous, and $Zm \cap Zh \in Z[I]$, so $m \in I$. Similarly n is continuous and $n \in I$. Since $f + m = g + n$ then $f \equiv_I g$.

8.8. **Remark.** This description of the congruence also holds for z-ideals in $C(X)$ [GJ, 5.4(a)]. It should be noted that Z_μ-ideals are partially ordered just as z-ideals by the method of [GJ, 5.2]. However it does not appear that the quotient semiring will be totally ordered in general. This is connected with the unsatisfactory behavior of the concept of prime ideal; the analogue of [GJ, 2.9] does not hold. The algebraic properties of the quotient semiring do behave properly for maximal m-ideals, as the following result shows; the quotients are m-*semifields*, in the sense that the nonzero elements are m-units.

8.9. Suppose I is a Z_μ-ideal and $f \in C_\mu(X)$ is an m-unit. Let $\phi: C_\mu(X) \to C_\mu(X)/I$. Then $\phi(f)$ is an m-unit.

Proof. Suppose $\phi(f)\,\phi(g) = \phi(f)\,\phi(h)$. Then by 8.7 $fg = fh$ on some $Zm \in Z[I]$. Since $Zf = \emptyset$ it follows that $g = h$ on Zm, and thus $\phi(g) = \phi(h)$.

8.10. If M is a maximal m-ideal then every nonzero element of $C_\mu(X)/M$ is an m-unit.

Proof. Let $\phi: C_\mu(X) \to C_\mu(X)/M$, and suppose $\phi(f) \neq 0$. Then $Zf \notin Z[M]$ and thus $Zf \cap Zg = \emptyset$ for some $g \in M$. Therefore $f + g$ is an m-unit, and since $\phi(f) = \phi(f+g)$, then $\phi(f)$ is an m-unit by 8.9.

8.11. It need not follow in an arbitrary semiring that 8.10 holds for a maximal m-ideal. Also $C_\mu(X)/M$ may have the property expressed even when M is not an m-ideal; for example, M may be a maximal ideal.

9. **The Wallman compactification.** The structure spaces of the semirings $C_\mu(X)$ eventually all agree and give the Wallman compactification, although as will be seen below they may give other compactifications for small ω_μ.

For each ω_μ let $\omega_\mu X$ be the maximal m-ideal space of $C_\mu(X)$ and define $\omega_\mu: X \to \omega_\mu X$ by $\omega_\mu(X) = p$, where p is the index of the maximal m-ideal M^x_μ.

9.1. The pair $(\omega_\mu X, \omega_\mu)$ is a compact extension of X and is a compactification if and only if fchr $X \leq \omega_\mu$.

Proof. To see that ω_μ is a map let $f \in C_\mu(X)$, so that $\mathfrak{M}_f$ is a basic closed set in $\omega_\mu X$. Then $\omega_\mu \leftarrow [\mathfrak{M}_f] = Zf$. Thus ω_μ is a map and moreover it is an embedding if and only if $Z_\mu(X)$ is a base for closed sets in X; that is, if and only if fchr $X \leq \omega_\mu$. Finally, $\omega_\mu[X]$ is dense in $\omega_\mu X$, since if $\omega_\mu[X] \subset \mathfrak{M}_f$ for some $f \in C_\mu(X)$ then $f = 0$, and thus $\mathfrak{M}_f = \omega_\mu X$, since 0 belongs to every ideal in $C_\mu(X)$.

A topological characterization of the pair $(\omega_\mu X, \omega_\mu)$ will now be given in terms of the theory of structures [HS3]. A filter γ on X is ω_μ-*cauchy* if it contains a member of every finite open cover of X by complements of sets in $Z_\mu(X)$. If λ is a filter on X the *zero hull* of λ is $[\lambda]_\mu = \{A \in \lambda$: there is $f \in C_\mu(X)$ with $Zf \in \lambda$ and $Zf \subset A\}$, and the cozero kernel of λ is $\langle\lambda\rangle_\mu = \{A \in \lambda$: there is $g \in C_\mu(X)$ with $X - Zg \in \lambda$ and $X - Zg \subset A\}$.

9.2. A filter γ on X is minimal ω_μ-cauchy if and only if it is the cozero kernel of same maximal Z_μ-filter.

Proof. Suppose γ is the cozero kernel of some maximal Z_μ-filter λ^p. Suppose $\bigcup (X - Zf_i) = X$; then $\bigcap Zf_i = \emptyset$ and thus some $Zf_i \notin \lambda^p$, so $Zf_i \cap Zg = \emptyset$ for some $Zg \in \lambda^p$, and then $X - Zf_i \in \gamma$. Therefore γ is ω_μ-cauchy. Now suppose $\nu \subset \gamma$ and ν is cauchy. If $A \in \gamma$ there are $f, g \in C_\mu(X)$ so that $Zf \subset X - Zg \subset A$ and $Zf \in \lambda^p$. Now since $\nu \subset \gamma \subset \lambda^p$ and $X - Zf \notin \lambda^p$ then $X - Zf \notin \nu$ and thus $X - Zg \in \nu$, so $A \in \nu$. Therefore γ is minimal ω_μ-cauchy.

Conversely suppose γ is a minimal ω_μ-cauchy filter. Set $\lambda = \{A$: there are $f_i \in C_\mu(X)$ with $X - Zf_i \notin \gamma$ and $\bigcap Zf_i \subset A\}$. Since γ is a cauchy filter it follows that λ is in fact a Z_μ-filter. Thus λ is in a maximal Z_μ-filter λ^p. Now if $A \in \langle\lambda^p\rangle_\mu$ then there are $f, g \in C_\mu(X)$ with $A \supset X - Zg \supset Zf \in \lambda^p$, and since $X - Zf \notin \gamma$ and γ is ω_μ-cauchy then $X - Zg \in \gamma$ and hence $A \in \gamma$. Thus $\langle\lambda^p\rangle_\mu \subset \gamma$. By the preceding paragraph $\langle\lambda^p\rangle_\mu$ is cauchy and since γ is minimal cauchy then $\langle\lambda^p\rangle_\mu = \gamma$.

9.3. The subsets $\omega_\mu[A]$ and $\omega_\mu[B]$ have disjoint closures in $\omega_\mu X$ if and only if A and B are separated by some $f \in C_\mu(X)$.

Proof. The sets $\mathfrak{M}_f = \{p: f \in M^p\} = \mathrm{cl}_{\omega_\mu X}(\omega_\mu[Zf])$ form a base for closed sets in $\omega_\mu X$. If A and B are separated by $f \in C_\mu(X)$ then by 7.9 they are contained in disjoint zero sets and thus their closures in $\omega_\mu X$ must be disjoint. Conversely if their closures are disjoint then since $\omega_\mu X$ is compact the closures must be contained in disjoint finite intersections of closed sets, so by 7.10 and 7.1 the given sets are contained in disjoint zero sets and hence are separated by some $f \in C_\mu(X)$ in view of 7.9.

9.4. The compactification $(\omega_\mu X, \omega_\mu)$ of X is the strict topological extension with the trace system of minimal cauchy filters on X.

Proof. It is clear from 9.3 that the trace filters are the cozero kernels of maximal Z_μ-filters, thus by 9.2 they are the minimal ω_μ-cauchy fiters. The extension is strict, since the collection of closures of zero sets is a base.

The *wallman character* of a space X, written wchr X, is defined as the least cardinal ω_μ such that every pair of disjoint closed sets are separated by some $f \in C_\mu(X)$. The Wallman compactification of X is described in [HS5], along with its properties useful in proving 9.6 below.

9.5. For each X, fchr $X \le$ wchr $X \le$ crdn X.

9.6. The pair $(\omega_\mu X, \omega_\mu)$ is the Wallman compactification $(\omega X, \omega)$ of X if and only if wchr $X \le \omega_\mu$.

Proof. If $(\omega_\mu X, \omega_\mu)$ is the Wallman compactification then by 9.3 it follows that wchr $X \le \omega_\mu$.

Conversely suppose wchr $X \le \omega_\mu$. Equality will be shown by examining the trace filters. The trace filters of the Wallman compactification are the open kernels of maximal closed filters, that is, the cozero kernels of maximal Z_ν-filters for large enough ω_ν, for example $\omega_\nu \ge$ crdn X. Given a maximal closed filter λ the assumed condition on separation of sets by $C_\mu(X)$ allows us to show that $[\lambda]_\mu$ is a maximal Z_μ-filter and that $\langle[\lambda]_\nu\rangle_\nu = \langle[\lambda]_\mu\rangle_\mu$, which is a trace filter for the compactification $(\omega_\mu X, \omega_\mu)$. On the other hand, given a maximal Z_μ-filter $\lambda^p = [\lambda^p]_\mu$ it belongs to some (unique, in fact) maximal closed filter $\lambda = [\lambda]_\nu$ and then $[\lambda]_\mu = \lambda^p$ and by the preceding argument $\langle\lambda^p\rangle_\mu = \langle\lambda\rangle_\nu$, which is a trace filter for the Wallman compactification.

9.7. **Remark.** The compactification $(\omega_\mu X, \omega_\mu)$ is the Shanin compactification [NG] corresponding to the base $Z_\mu(X)$ for closed sets (when fchr $X \le \omega_\mu$). Every Shanin compactification may in fact be obtained as the structure space of an appropriate subsemiring of some $C_\mu(X)$; $\omega_\mu \ge$ crdn X will suffice.

9.8. **Counterexample.** The compactification $(\omega_\mu X, \omega_\mu)$ is not described uniquely by condition 9.3, even among strict compactifications. In fact a compact space may be densely embeddable in another compact space satisfying that condition, even for $\omega_\mu \geq \text{crdn } X$.

9.9. **Counterexample.** The wallman character may be strictly larger than the fréchet character. Consider the Tychonoff plank $\mathbf{T}$ [GJ, 8.20]. It has fréchet character 2, since the spaces $\mathbf{N}^*$ and $\mathbf{W}^*$ each have a base of open-and-closed subsets. Now it can readily be shown by standard techniques that the compactification $(\omega_0\mathbf{T}, \omega_0)$ is the space $\mathbf{W}^* \times \mathbf{N}^*$, and that every $f \in C_0(\mathbf{T})$ has a unique extension to $\omega_0 f \in C_0(\omega_0\mathbf{T})$. The compactification $\omega_0\mathbf{T}$ is therefore not the Wallman compactification of $\mathbf{T}$; it is in fact the Stone-Čech compactification. Since $\text{crdn } \mathbf{T} = \omega_1$ it follows readily that the Wallman compactification is the pair $(\omega_1 X, \omega_1)$.

9.10. If X is compact and $\text{fchr } X$ is infinite then $\text{fchr } X = \text{wchr } X$.

Proof. One need only observe that for a compact space disjoint closed sets must be contained in disjoint members of any base that is closed under finite intersection.

9.11. For each space X with $\text{fchr } \omega X$ infinite, $\text{wchr } X \leq \text{fchr } \omega X$.

An ideal I in $C_\mu(X)$ if *fixed* if $I \subset M_\mu^x$ for some $x \in X$. The following result is easily shown.

9.12. Let $\text{fchr } X \leq \omega_\mu$. Then X is compact if and only if every (maximal) m-ideal in $C_\mu(X)$ is fixed.

As an immediately corollary the following characterization theorem for compact spaces can be given.

9.13. Let $\text{fchr } X \leq \omega_\mu$ and $\text{fchr } Y \leq \omega_\mu$, where X and Y are compact. Then X and Y are homeomorphic if and only if the semirings $C_\mu(X)$ and $C_\mu(Y)$ are isomorphic.

10. **ω_μ-entire and ω_μ-total spaces.** In the theory of the rings $C(X)$ the real maximal ideals, which are the maximal ideals M for which $C(X)/M$ is the real field, are a very important class of ideals. In the rings $C_\mu(X)$, a similar role is played by the ω_μ-entire ideals.

Let M be a maximal m-ideal in $C_\mu(X)$, with quotient semiring Q and projection map $\Phi: C_\mu(X) \rightarrow Q$. If $f, g \in C_\mu(X)$ are constant functions with $f \neq g$, then it follows from 8.7 that $\Phi(f) \neq \Phi(g)$; thus the semiring Q contains a copy of $\mathbf{F}_\mu$ as the images of the constant functions. An *ω_μ-entire* ideal is a maximal m-ideal M for which this copy of $\mathbf{F}_\mu$ is the entire semiring Q; that is, every function is congruent to a constant function. The space X is *ω_μ-entire* if every ω_μ-entire ideal is fixed, and the space X is *ω_μ-total* if every maximal m-ideal in $C_\mu(X)$ is ω_μ-entire.

If $f: X \rightarrow Y$ is a map and M is a maximal m-ideal define $f^\#(M) = \{g \in C_\mu(Y): gf \in M\}$.

10.1. If M is ω_μ-entire then $f^{\#}(M)$ is ω_μ-entire.

Proof. Suppose $q \in C_\mu(Y)$; then $gf \in C_\mu(X)$ and thus since M is ω_μ-entire there is $\delta \in \mathbf{F}_\mu$ with $(gf)^{\leftarrow}(\delta) \in Z[M]$ (using 8.7). Thus $g^{\leftarrow}(\delta) \in Z[f^{\#}(M)]$. It follows that if Zg intersects $Z[f^{\#}(M)]$ then $\delta = 0$ and so $g \in f^{\#}(M)$. Therefore $f^{\#}(M)$ is a maximal Z_μ-ideal and is ω_μ-entire.

10.2. **Remark.** The analogue of 10.1 also holds in $C(X)$; the image of a real maximal ideal under the sharp mapping is a real maximal ideal. This allows the improvement of a few proofs in [GJ]; for example, [GJ, Lemma 8.12] is not needed for the alternative proof of [GJ, Theorem 8.11].

10.3. Let fchr $X \leq \omega_\mu$. If $A \subset X$ is closed and X is ω_μ-entire then A is ω_μ-entire.

Proof. Suppose M is a ω_μ-entire ideal on A and let $f: A \to X$ be the inclusion map. Then $f^{\#}(M)$ is ω_μ-entire, so it is equal to M^x for some $x \in X$, in view of the assumed ω_μ-entireness of X. Since A is closed it follows that $x \in A$ and $M = M^x$.

10.4. Let fchr $X_\alpha \leq \omega_\mu$ for each $\alpha \in \mathfrak{k}$. Then $X = \Pi_{\alpha \in \mathfrak{k}} X_\alpha$ is ω_μ-entire if and only if each factor is ω_μ-entire.

Proof. Suppose each factor is ω_μ-entire. Let M be a ω_μ-entire ideal on X. For each $\alpha \in \mathfrak{k}$ let $\Pi_\alpha: X \to X_\alpha$ be the projection. Since each X_α is ω_μ-entire it follows that for each $\alpha \in \mathfrak{k}$ there is $x(\alpha) \in X_\alpha$ with $\Pi^{\#}_\alpha(M) = M^{x(\alpha)}$. Define $x \in X$ by $\Pi_\alpha(x) = x(\alpha)$. Suppose $B \subset X$ is closed with $x \notin B$. Then there are closed sets $B_\alpha \subset X_\alpha$ with $B \subset C = \bigcup \Pi^{\leftarrow}_\alpha [B_\alpha]$ and $x \notin C$, for some finite set of indices α. Now $\Pi^{\leftarrow}_\alpha [B_\alpha] \in Z[M]$ for each α, since from $x \notin C$ there follows $x(\alpha) = \Pi_\alpha(x) \notin B_\alpha$ and thus $B_\alpha \in Z[\Pi^{\#}_\alpha(M)]$. Thus $C \notin M$ (since maximal m-ideals are prime) and thus $B \notin M$. It follows that $M = M^x$.

Conversely if X is ω_μ-entire then each factor is ω_μ-entire, since it is homeomorphic to a slice in X, which will be closed.

10.5. Let $\{Y_\alpha\}_{\alpha \in \mathfrak{k}}$ be a collection of ω_μ-entire subspaces of Y, where fchr $Y \leq \omega_\mu$. Then $Z = \bigcap Y_\alpha$ is ω_μ-entire.

Proof. Let M be an ω_μ-entire ideal in $C_\mu(Z)$. By 10.1 $j^{\#}_\alpha(M)$ is ω_μ-entire for each $\alpha \in \mathfrak{k}$, where $j_\alpha: Z \to Y_\alpha$ is the inclusion. Thus $j^{\#}_\alpha(M) = M^{y(\alpha)}$ for some $y(\alpha) \in Y_\alpha$, since Y_α is ω_α-enitre. Now if $\alpha, \beta \in \mathfrak{k}$ and $f \in C_\mu(Y)$ then $j^{\leftarrow}f^{\leftarrow}(f(y(\alpha))) \in M$ and $j^{\leftarrow}f^{\leftarrow}(f(y(\beta))) \in Z[M]$, where $j: Z \to Y$ is the inclusion. Therefore, $f(y(\alpha)) = f(y(\beta))$ for each $\alpha, \beta \in \mathfrak{k}$ and thus $y(\alpha) = y(\beta)$ for each $\alpha, \beta \in \mathfrak{k}$; the common value y is a point y of Z and clearly $M = M^y$.

10.6. A compact space is ω_μ-entire for every ω_μ.

10.7. Any space X is ω_μ-entire for $\omega_\mu \geq$ crdn X.

Proof. When $\omega_\mu \geq$ crdn X the partition of X by singletons is a ω_μ-partition. Taking a function with this as associated partition it follows that an ω_μ-entire ideal musṭ contain a singleton member.

10.8. **Remark.** In accordance with the above result the interest in ω_μ-entireness lies in examining the property for relatively low valued cardinals. It is clear from 10.3 that spaces with very large cardinals may be ω_μ-entire for very low ω_μ.

10.9. **Counterexample.** The space $\mathbf{W}$ of [GJ, 5.12] is not ω_0-entire. There is precisely one free maximal Z_0-filter on $\mathbf{W}$, generated by cofinal closed subsets of $\mathbf{W}$. Given any countable partition of $\mathbf{W}$ one of the sets is cofinal, and it follows that the free filter is ω_0-entire.

Since ω_μ-entireness is defined by only formal analogy with realcompactness, it is rather surprising that the concepts bear essentially the same relation to discreteness. Write $\mathbf{D}_\nu$ for the discrete space of cardinal ω_ν.

10.10. If ω_μ is nonmeasurable then $\mathbf{D}_\nu$ is ω_μ-entire for every ω_μ.

10.11. If ω_ν is measurable then $\mathbf{D}_\nu$ is not ω_μ-entire for any nonmeasurable ω_μ.

Proofs. Suppose $\mathbf{D}_\nu$ is not ω_μ-entire. Then there is an ω_μ-entire filter M on $\mathbf{D}_\nu$ that is not fixed. Since $\mathbf{D}_\nu$ is discrete, then $\mathbf{Z}_\mu(\mathbf{D}_\nu)$ is just the collection of subsets of $\mathbf{D}_\nu$ and thus M is an ultrafilter. Then by [GJ, 12.2] there is a nonzero finitely additive $\{0, 1\}$-valued set function associated with M. Since M is ω_μ-entire and hence ω_0-entire, it follows that this set function is actually a measure on $\mathbf{D}_\nu$. Since M is not fixed, it follows that $\omega_\mu = \operatorname{crdn} \mathbf{D}_\nu$ is measurable.

Conversely if ω_ν is measurable then there is a free measure on $\mathbf{D}_\nu$ and the associated ultrafilter M is a free maximal Z_μ-filter for every ω_μ. If ω_μ is nonmeasurable then the measure is ω_μ-additive by [GJ, 12.3(a)] and it follows that M is ω_μ-entire; thus $\mathbf{D}_\nu$ is not ω_μ-entire.

We now examine the ω_μ-total spaces, which possess some properties analogous to those of pseudocompact spaces.

10.12. If $\operatorname{fchr} X \leq \omega_\mu$ then X is compact if and only if X is ω_μ-entire and ω_μ-total.

A space X is (ω_ν, ω_μ)-compact if every open cover of cardinal $< \omega_\nu$ has a subcover of cardinal $< \omega_\mu$; for example, the countably compact spaces are the (ω_1, ω_0)-compact spaces in this sense.

10.13. An $(\omega_{\mu+1}, \omega_0)$-compact space is ω_μ-total.

Proof. Suppose X is not ω_μ-total and λ is a maximal Z_μ-filter such that $M = I[\lambda]$ is not ω_μ-entire. Then there is an ω_μ-partition $\{B_\delta\}_{\delta<\omega_\mu}$ of X such that λ contains no B_δ. It follows that each B_δ is disjoint from some $C_\delta \in Z_\mu(X)$ such that $C_\delta \in \lambda$. Now $\bigcap_{\delta<\omega_\mu} C_\delta = \Phi$ since $\bigcup_{\delta<\omega_\mu} B_\delta = X$, yet no finite intersection of the C_δ is empty. It follows that the cover $\{X - C_\delta\}_{\delta<\omega_\mu}$ of X has no finite subcover.

11. **The extension $(\upsilon_\mu X, \upsilon_\mu)$.** The process for $C_\mu(X)$ analogous to the formation of the Hewitt realcompactification for $C(X)$ assumes an even greater importance in view of the nonuniqueness of extension of functions to $\omega_\mu X$.

11.1. Every fixed maximal m-ideal in $C_\mu(X)$ is ω_μ-entire.

Letting $v_\mu X$ be the subspace of $\omega_\mu X$ whose points correspond to ω_μ-entire ideals it follows from 11.1 that the inclusion $\omega_\mu: X \to \omega_\mu X$ restricts to v_μ: $X \to v_\mu X$. Also for each $p \in v_\mu X$ and $f \in C_\mu(X)$ define $v_\mu(f) = \delta$, where $f^{\leftarrow}(\delta) \in M^p$.

11.2. For each ω_μ the function $f \to v_\mu(f)$ is an isomorphism of $C_\mu(X)$ onto $C_\mu(v_\mu X)$, and $v_\mu(f)$ is the unique $g \in C_\mu(v_\mu X)$ for which $g v_\mu = f$.

Proof. To see that $v_\mu(f)$ is a map note that $v_\mu(f)^{\leftarrow}(\delta) = \{p \in v_\mu X: f^{\leftarrow}(\delta) \in M^p$ $= \mathrm{cl}_{v_\mu X}\ v_\mu[f^{\leftarrow}(\delta)]$; thus by 3.1 the function $v_\mu(f)$ is a map. Also clearly $v_\mu(f)\, v_\mu = f$ and $f \to v_\mu(f)$ is one-one. Now suppose $g \in C_\mu(v_\mu X)$ and $f = g\, v_\mu$. If $p \in v_\mu X$ then $f^{\leftarrow}(\delta) \in M^p$ for some δ and it follows that $p \in \mathrm{cl}_{v_\mu X}\ v_\mu[f^{\leftarrow}(\delta)]$. If $g(p) \neq \epsilon$ then $g^{\leftarrow}[\mathrm{F}_\mu - \{\epsilon\}]$ is a neighborhood of $p \in v_\mu X$. Thus $g^{\leftarrow}[\mathrm{F}_\mu - \{\epsilon\}] \cap v_\mu[f^{\leftarrow}(\delta)] \neq \emptyset$ from which it follows that $\epsilon \neq \delta$. Thus $g(p) = v_\mu(f)(p)$ for each $p \in v_\mu X$.

11.3. The space $v_\mu X$ is ω_μ-entire.

Proof. The isomorphism constructed in 11.2 preserves constant functions and can readily be shown to preserve ω_μ-entire ideals in either direction. It follows easily that the ω_μ-entire ideals on $v_\mu X$ are fixed.

11.4. The space X is ω_μ-entire if and only if $v_\mu[X] = v_\mu X$.

11.5. The space X is ω_μ-total if and only if $v_\mu X = \omega_\mu X$.

11.6. Let $\mathrm{fchr}\ X \leq \omega_\mu$. Then $\mathrm{fchr}\ v_\mu X = \mathrm{fchr}\ X$.

Proof. If $\mathrm{fchr}\ X \leq \omega_\mu$ then by 9.1 the map v_μ is an embedding of X into $v_\mu X$, from which it follows by 3.9 that $\mathrm{fchr}\ X \leq \mathrm{fchr}\ v_\mu X$. To see that $\mathrm{fchr}\ v_\mu X \leq \mathrm{fchr}\ X$ note that the closures in $v_\mu X$ of zero sets of functions in $C_\mu(X)$ form a closed base for $v_\mu X$, and as established in 11.2 these are the zero sets of functions in $C_\mu(v_\mu X)$.

The most useful property of $v_\mu X$ is its functorial property which is now developed.

11.7. Let $\mathrm{fchr}\ X \leq \omega_\mu$, $\mathrm{fchr}\ Y \leq \omega_\mu$. Then every $f: X \to Y$ has a unique extension $v_\mu(f): v_\mu X \to v_\mu Y$.

Proof. For each $p \in v_\mu X$ the ideal $f^{\#}(M^p)$ is an ω_μ-entire ideal N^q in $C_\mu(Y)$. Define $v_\mu(f): v_\mu X \to v_\mu Y$ by $v_\mu(f)(p) = q$ where $N^q = f^{\#}(M^p)$. Since $Z_\mu[v_\mu Y]$ is a base for closed sets in $v_\mu Y$ by 11.6, then it suffices to show that $g\, v_\mu(f) \in C_\mu(v_\mu X)$ whenever $g \in C_\mu(v_\mu Y)$. To see this it suffices to show that $g\, v_\mu(f) = v_\mu(hf)$, where $h = g\, v_\mu \in C_\mu(Y)$. This latter relation is readily shown.

11.8. The category of ω_μ-entire spaces is an epireflective subcategory of the category of spaces having fréchet character $\leq \omega_\mu$, via the ω_μ-entire extension $X \to v_\mu X$.

11.9. **Remark.** The epireflective property of the class of ω_μ-entire spaces could have been used to establish 10.3, 10.4, and 10.5. The epireflection $\upsilon_\mu X$ can be constructed by taking the evaluation embedding [WD] of X into $\mathbf{F}_\mu^{C_\mu(X)}$ and finding $\upsilon_\mu X$ as the intersection of all ω_μ-entire subspaces of the evaluation product that contains X.

11.10. **Remark.** Although the class of ω_μ-entire spaces is productive and closed hereditary it seems unlikely that the class is generated as the closed subspaces of products of a single member, as a Herrlich-Mrówka class of compactness [HH]. The space $\mathbf{F}_\mu$ certainly does not generate the class in this fashion. The analogue of Husek's universal space [HK] used to characterize Herrlich's k-compact spaces is the product of ω_μ copies of $\mathbf{F}_\mu$ with a point deleted; this is a universal space for a class other than that of ω_μ-entire spaces.

12. **Compact-small supports.** A subset V of X is *compact-small* if when A is closed in X and $A \subset V$ then A is compact. The space X is *locally compact-small* if every point has a compact-small neighborhood. As is shown in [HS4] a locally compact space is locally compact-small, and a locally compact-small regular space is locally compact, but a locally compact-small T_2 space need not be locally compact. It is also shown in [HS4] that a space is open in its Wallman compactification if and only if it is locally compact-small.

The cozero set of $f \in C_\mu(X)$ is the set $Sf = X - Zf$. The set of functions with compact-small cozero set is written $C_\mu^s(X)$.

12.1. Let $\omega_\mu \geq \text{wchr } X$. Then $C_\mu^s(X)$ is the intersection of all free maximal m-ideals.

Proof. Suppose $f \in C_\mu(X)$ and $f \notin M$, where M is a free maximal m-ideal. Then $Zf \cap Zg = \emptyset$ for some $g \in M$. Then Zg is not compact since M is free, and it follows that $f \notin C_\mu^s(X)$. Conversely suppose $f \notin C_\mu^s(X)$; then there is a noncompact closed set $B \subset Sf$ and since $\omega_\mu \geq \text{wchr } X$ there is $g \in C_\mu(X)$ with $B \subset Zg \subset Sf$. Since Zg is noncompact, g belongs to some free maximal m-ideal M, and certainly $f \notin M$.

12.2. **Remark.** When $\omega_\mu < \text{wchr } X$ then 12.1 may not hold, as may be seen by considering $C_0(\mathrm{T})$ and $C_0^s(\mathrm{T})$ where T is the Tychonoff plank [GJ, 8J]. However, in this case one replaces $C_\mu^s(X)$ by the functions $f \in C_\mu(X)$ such that any zero set contained in Sf is compact. This is precisely analogous to [GJ, 4E]; such functions are said to have *small support.*

The collection $C_\mu^s(X)$ is an ideal in $C_\mu(X)$ but may not be a subsemiring since it may have no multiplicative unit. Just as in $C(X)$ one considers the subsemiring $C_\mu^c(X)$ of functions that are constant on the complement of some compact-small set.

12.3. The space X has the weak topology of $C_\mu^c(X)$ if and only if it is locally compact-small and $\text{fchr } X \leq \omega_\mu$.

Proof. Suppose X has the weak topology of $C^c_\mu(X)$. If $x \in X$ and for some $f \in C_\mu(X)$ and some $\delta \neq f(x)$ the set $X - f^{\leftarrow}(\delta)$ is compact-small, then x has a compact-small neighborhood. If this does not occur for some $x \in X$ then $X - f^{\leftarrow}(f(x))$ is compact-small for each $f \in C_\mu(X)$. In this case consider any open cover α of X. There is $V \in \alpha$ with $x \in V$ and thus there are $f_1, \cdots, f_n \in C_\mu(X)$ and $\delta_1, \cdots, \delta_n \in \mathrm{F}_\mu$ with $x \in \bigcap (X - f_i^{\leftarrow}(\delta_i)) \subset V$. Now it follows that $X - V \subset \bigcup (X - f_i^{\leftarrow}(f_i(x)))$, so $X - V$ is compact, and thus α has a finite subcover and therefore X is compact, and thus certainly locally compact-small.

Now suppose X is locally compact-small, and fchr $X \leq \omega_\mu$. Let $x \in X$ and suppose V is a compact-small neighborhood of x. There is $f \in C_\mu(X)$ with $x \in X - Zf \subset V$, and thus $f \in C^s_\mu(X) \subset C^c_\mu(x)$. It follows that X has the weak topology of $C^c_\mu(X)$.

12.4. **Remark.** The m-units in $C^c_\mu(X)$ are the members with empty zero set, since $f_1 \in C^c_\mu(X)$ if and only if $f \in C^c_\mu(X)$, so that the proof of 7.3 applies in $C^c_\mu(X)$. It then follows as noted in 8.4 that there is a bijection between maximal m-ideals in $C^c_\mu(X)$ and maximal filters in the lattice $Z[C^c_\mu(X)]$. It can then be shown that the Alexandrov one-point compactification of X is the structure space of $C^c_\mu(X)$ when X has the weak topology of $C^c_\mu(X)$. Further details are given in [HS4]. Note that since $Z[C^c_\mu(X)] = Z[C^s_\mu(X)]$, a space has the weak topology of $C^c_\mu(X)$ if and only if it has the weak topology of $C^s_\mu(X)$.

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